

Geometric phase for degenerate states of spin-1 and spin-1/2 pair

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The geometric phase of a bi-particle model is discussed. One can drive the system to evolve by external magnetic field, thereby controlling the geometric phase. The relationship between the geometric phase and the structures of the initial state is obtained. At last we extend the results to a more general case.

PACS numbers: 03.65.Vf, 03.65.Yz

Keywords: Geometric Phase, dynamical evolution

I. INTRODUCTION

Geometric phase (GP) in quantum theory attracted great interest since Berry [1] showed that the state of quantum system acquires a purely geometric feature in addition to the usual dynamical phase when it is varied slowly and eventually brought back to its initial form. The general to nonadiabatic evolution extension was formulated by Aharonov and Anandan [2]. The extension to noncyclic evolution was done by Samuel and Bhandara [3].

Uhlmann [4] was the first to introduce the notion of GP for mixed state. By considering a purification and the notion of parallelity, he furnished a definition of GP for mixed states. Later, Sjöqvist *et al.* [5] introduced a formalism that defines the mixed state GP with the experimental context of quantum interferometry. This definition was verified experimentally later [6]. Tong *et al.* gave a kinematic approach to define GP in mixed states undergoing unitary evolution [7]. The kinematic approach was also used to study the off-diagonal geometric phases for mixed state, including nondegenerate and degenerate cases [8]. The off-diagonal mixed state geometric phase could contain interference information when the diagonal phase is undefined.

GP of composite system has drawn much attention during recent years. X. X. Yi *et al.* studied the geometric phase of composite mixed state [9]. The connection between the GP and quantum phase transition of many-body system was studied [10]. GP can be used to detect the quantum phase transition points [11]. The scaling behavior of GP in the vicinity of the quantum phase transition point were also discussed [12].

On the other hand, GP for some system driven by external fields is discussed [13, 14, 15, 16, 17, 18]. In Ref.[19], the authors calculated the geometric phase of a two-level system driven by a quantized magnetic field subject to phase dephasing and found that the phase reduces to the standard GP in the weak-coupling limit.

GPs are interesting both from a fundamental point of view and for their applications. Due to the fact that GP depend only on some global geometric properties, geometric quantum computation is one of the most important [20, 21]. The geometric quantum gate was shown robust against decoherence [22] and has built-in fault-tolerant features.

In this paper, the GP for one two-interaction-spin model which is composed of spin-1 and spin-1/2 under unitary evolvment will be discussed. GP is dependent upon the initial state and the Hamiltonian. In Sec.II, GP for different structure of initial states is given. The effect of magnetic field is discussed. In Sec.III, GP of a more general case will be considered.

II. GEOMETRIC PHASE FOR SPIN-1 AND SPIN-1/2 PAIR

We consider a simple system of spin-1 and spin-1/2 with anisotropic Heisenberg coupling in an uniform magnetic field as follows

$$H = H_0 + H_1 = \frac{J}{2} (\sigma_x \cdot S_x + \sigma_y \cdot S_y + \Delta \sigma_z \cdot S_z) + B(\frac{1}{2}\sigma_z + S_z), \quad (1)$$

where σ 's refer to the Pauli matrices for spin-1/2 and S ' denote the spin operators for spin-1, Δ is the anisotropy factor. It is obvious that the commutator $[H_0, H_1] = 0$. As one knows, in the ${}^6\text{Li}$ atom, the nucleus has spin-1 and the electrons have total spin-1/2. This can be regarded as an example. Throughout this paper, the spin-1/2 states are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$ while the spin-1 states are denoted by $|\uparrow\rangle$, $|0\rangle$, $|\downarrow\rangle$. Then the evolvment matrix $U(t) = \exp(-itH)$, where the Planck constant \hbar is set to one. At time t , the density matrix is described by $\rho(t) = U(t)\rho(0)U(t)^\dagger$.

We assume at the initial time $t = 0$, the magnetic field is absent. The initial state is the ground state of H_0 . Then the magnetic field is imposed to drive the system to evolve. In the following, we will study the different cases: $J > 0$ and $J < 0$.

A. $J > 0$

One can easily obtain the eigenvectors and eigenvalues of the Hamiltonian H_0 . We assume at the initial time $t = 0$, the magnetic field B is absent, i.e., $B = 0$. In this case, the coupling constant J is positive, so the ground state energy is that $\Delta J/2$ when $\Delta < -1$ and $(-\Delta - \sqrt{8 + \Delta^2})J/4$ when $\Delta > -1$. At the point $\Delta = -1$, it is a critical point separating two different structures of ground states. In details, when $\Delta < -1$, there are two corresponding eigenvectors: $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. When $\Delta > -1$, the ground state is also twofold degenerate and the eigenvectors are:

$$|\Psi_1\rangle = \frac{1}{F_+} \left(|\downarrow 0\rangle - \frac{\Delta + \sqrt{\Delta^2 + 8}}{2\sqrt{2}} |\uparrow\downarrow\rangle \right), \quad (2)$$

$$|\Psi_2\rangle = \frac{1}{F_-} \left(|\downarrow\uparrow\rangle + \frac{\Delta - \sqrt{\Delta^2 + 8}}{2\sqrt{2}} |\uparrow 0\rangle \right), \quad (3)$$

where F_{\pm} is the normalized factor.

In the case $\Delta < -1$, the two eigenvectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ span a 2-dimensional eigenspace. Here the initial state is assumed to be a pure state, i.e.,

$$|\psi(t=0)\rangle = \cos\theta |\uparrow\uparrow\rangle + \sin\theta e^{i\phi} |\downarrow\downarrow\rangle. \quad (4)$$

The density matrix of the initial state $\rho_0 = |\psi(0)\rangle\langle\psi(0)|$ has $[\rho_0, H_0] = 0$, so the state will not evolve under the Hamiltonian H_0 . GP vanishes. In order to drive the system to evolve and control the GP, one can subject external magnetic field \mathbf{B} to the system at the time $t = 0^+$. It is obvious that when the external magnetic field is added, no matter how weak it is, the degeneracy of the ground state is destroyed, due to the Zeemann split. Our interest is to study the evolution of the initial state $|\psi(t=0)\rangle$ under the new Hamiltonian $H = H_0 + H_1$. It is obvious that $[\rho_0, H] \neq 0$. After a cyclic evolution, $\rho(T) = \rho(0)$, then one has

$$T_1 = \frac{2n\pi}{3B}, \quad n \in \mathbf{Z}. \quad (5)$$

It is the function the external magnetic field. In the following we restrict ourselves to $n = 1$ and from this on. As one knows, for a pure state, the GP can be defined as

$$\gamma_G[U] = \gamma_t - \gamma_d = \arg\{\langle\psi(0)|U(\tau)|\psi(0)\rangle\} + i \int_0^\tau \langle\psi(0)|U(t)^\dagger \dot{U}(t)|\psi(0)\rangle dt. \quad (6)$$

The first term on the right side of Eq.6 is the total phase and the second term corresponds to the dynamical phase. One can obtain that

$$\gamma_G = \gamma_t - \gamma_d = 2\pi \cos^2 \theta. \quad (7)$$

For $\Delta > -1$, the initial state is assumed to be

$$|\Psi(t=0)\rangle = \cos\theta |\Psi_1\rangle + \sin\theta e^{i\phi} |\Psi_2\rangle. \quad (8)$$

The period $T_2 = 2\pi/B$. One can obtain the GP is

$$\gamma_G = \gamma_T - \gamma_d = 2\pi \sin^2 \theta. \quad (9)$$

At the point $\Delta = -1$, the universal pure state spanned by the four eigenvectors is given by

$$|\Psi\rangle = \sin\theta_1 \sin\theta_2 \cos\theta_3 |\Psi_1\rangle + \sin\theta_1 \sin\theta_2 \sin\theta_3 e^{i\phi_1} |\Psi_2\rangle + \sin\theta_1 \cos\theta_2 e^{i\phi_2} |\uparrow\uparrow\rangle + \cos\theta_1 e^{i\phi_3} |\downarrow\downarrow\rangle. \quad (10)$$

One can know the period $T_3 = 2\pi/B$ then the GP is obtained that

$$\gamma_G = \pi(1 - 3\cos^2\theta_1 + \sin^2\theta_1(3\cos^2\theta_2 - \cos 2\theta_3 \sin^2\theta_2)). \quad (11)$$

We only consider some simple cases, for example when $\theta_1 = \tan^{-1} \sqrt{3}$, $\theta_2 = \tan^{-1} \sqrt{2}$ and $\theta_3 = \pi/4$, which means the absolute value of the coefficients of all superposed states are $1/2$, the GP $\gamma_G = \pi$.

As we know, in realistic world, the system is subject to interact with the environment inevitably. Due to the effect of decoherence, the state will become mixed. The off-diagonal terms approach zero. Before the external magnetic field is imposed, one can assume that the initial state is mixed. When $\Delta < -1$, the density matrix is assumed to be

$$\rho(0) = a |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + (1-a) |\downarrow\downarrow\rangle\langle\downarrow\downarrow|. \quad (12)$$

Here $0 \leq a \leq 1$. In such a space, the density matrix is

$$\rho(0) = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix}, \quad (13)$$

the unitary operator in such a subspace is

$$U(t) = \begin{pmatrix} \exp(-i\frac{3B+\Delta J}{2}t) & 0 \\ 0 & \exp(i\frac{3B-\Delta J}{2}t) \end{pmatrix}. \quad (14)$$

Using the definition of GP of degenerate mixed state given in Ref. [9], one can know the GP $\gamma_G = 0$ (in Sec.III more details will be given). In fact ρ_0 and the Hamiltonian H are mutual commutative, the initial state will not varied with the time. No matter what the coefficients are, in this case the GP still keeps zero.

When $\Delta > -1$, the density matrix is assumed to be

$$\rho(0) = b |\Psi_1\rangle\langle\Psi_1| + (1-b) |\Psi_2\rangle\langle\Psi_2|. \quad (15)$$

At the point $\Delta = -1$, the density matrix is

$$\rho(0) = p_1 (|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + p_2 |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + p_3 |\Psi_1\rangle\langle\Psi_1| + (1-p_1-p_2-p_3) |\Psi_2\rangle\langle\Psi_2|). \quad (16)$$

In the same way, the GPs of the above states are that $\gamma_G = 0$. From above calculations, for the initial mixed state of this model, the GP remains zero independent on the values of the coefficients.

B. $J < 0$

In this subsection, we will take into account the case in which $J < 0$. The Hamiltonian can be rewritten as

$$H = H_0 + H_1 = \frac{-|J|}{2} (\sigma_x \cdot S_x + \sigma_y \cdot S_y + \Delta \sigma_z \cdot S_z) + B(\frac{1}{2}\sigma_z + S_z), \quad (17)$$

Before the initial time $t = 0$, the magnetic field is absent. The critical point here is $\Delta_C = 1$. When $\Delta < 1$, the ground state energy is $(-\Delta + \sqrt{8 + \Delta^2})J/4$. It is still twofold degenerate and there are two different eigenvectors:

$$|\phi_1\rangle = \frac{1}{N_1} \left(\frac{\sqrt{8 + \Delta^2} - \Delta}{2\sqrt{2}} |\uparrow\downarrow\rangle + |\downarrow 0\rangle \right), \quad (18)$$

$$|\phi_2\rangle = \frac{1}{N_2} \left(\frac{\sqrt{8 + \Delta^2} + \Delta}{2\sqrt{2}} |\uparrow 0\rangle + |\downarrow\uparrow\rangle \right), \quad (19)$$

where $N_{1,2}$ are the normalization factors. The initial state is assumed to be pure,

$$|\Psi(0)\rangle = \cos\theta |\phi_1\rangle + \sin\theta e^{i\phi} |\phi_2\rangle. \quad (20)$$

In the same way as in Sec.II A, one can obtain the GP as

$$\gamma_G = \gamma_t - \gamma_d = 2\pi \sin^2 \theta. \quad (21)$$

When $\Delta > 1$, the ground state energy is $J\Delta/2$, the two corresponding eigenvectors are $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. The initial state is assumed to be pure then

$$|\psi(t=0)\rangle = \cos\theta |\uparrow\uparrow\rangle + \sin\theta e^{i\phi} |\downarrow\downarrow\rangle. \quad (22)$$

One can easily obtain the GP is as the same as Eq.7: $\gamma_G = 2\pi \cos^2 \theta$.

At the critical point $\Delta = 1$, the state is fourfold degenerate. In the same way, when the initial state is pure, GP will have the same form as Eq.11.

For mixed stats, repeating the discussion in subsection.II A, one can know the GP remains zero for the above three cases.

III. DISCUSSION AND SUMMARY

In this section, one can consider a general model, $H = H_0 + H_1$, in which H_0 and H_1 are arbitrary but $[H_0, H_1] = 0$. Therefore, one has $e^{-iHt} = e^{-iH_0t}e^{-iH_1t}$. It is assumed that before $t = 0$, one has $H = H_0$. The ground-state energy is degenerate and the corresponding states are $|\phi_i\rangle$, $i = 1 \dots n$. The initial state is assumed to be the linear superposition of the eigenstates, i.e., $|\psi(0)\rangle = \sum_i c_i |\phi_i(0)\rangle$, $\sum_i |c_i|^2 = 1$. Then $\rho_0 = |\psi(0)\rangle\langle\psi(0)|$. Owing to the fact $[\rho_0, H_0] = 0$, the initial state does not evolve with the time. One can disturb the system by adding H_1 and $[\rho_0, H_1] \neq 0$. After a cyclic evolution, one can obtain the reduced GP:

$$\gamma_G = \arg\langle\Psi_0|e^{-iH_1T}|\Psi_0\rangle + \langle\Psi_0|H_1|\Psi\rangle T. \quad (23)$$

The period T satisfies $[e^{-iH_1T}, \rho_0] = 0$. The evolution is only dependent on the H_1 term, and the initial state is dependent only on the H_0 terms.

In realistic world, the interactions with the system and the environment always reduce the coherence of the system. The off-diagonal elements of the density matrix of the system approaches zero. So one can assume the initial state is mixed, $\tilde{\rho}_0 = \sum_i p_i |\phi_i(0)\rangle\langle\phi_i(0)|$, where $\sum_i p_i = 1$. The corresponding eigenenergy is E_0 , i.e., $H_0|\phi_i(0)\rangle = E_0|\phi_i(0)\rangle$.

The off-diagonal geometric phase factor of the degenerate mixed state is defined as [7]

$$\gamma_{\rho_{j_1} \dots \rho_{j_l}}^{(l)} = \Phi \left[\text{Tr} \left(\prod_{\alpha=1}^l U(\tau) V_{j_\alpha}^\parallel(\tau) \sqrt{\rho_{j_\alpha}(0)} \right) \right]. \quad (24)$$

where $\Phi[z] = z/|z|$. If $l = 1$, this reduces to the diagonal geometric phase factor and, if $l = 2$, we obtain the off-diagonal pure state geometric phase. The eigenspace is spanned by the basis $|\phi_k\rangle$. Here we are only to study $l = 1$ order GP,

$$U(T) = e^{-i(H_0+H_1)T} = e^{-iE_0T} e^{-iH_1T}, \quad (25)$$

$$V^\parallel = e^{iE_0T} \sum_i e^{iT\langle\phi_k|H_1|\phi_k\rangle} |\phi_k\rangle\langle\phi_k|, \quad (26)$$

so that one can know the GP factor is

$$\gamma = e^{i\gamma_G} = \Phi \left[\sum_k p_k \langle\phi_k| e^{-iH_1T} |\phi_k\rangle e^{iT\langle\phi_k|H_1|\phi_k\rangle} \right]. \quad (27)$$

It is clear when H_1 is diagonal in basis $|\phi_k\rangle$, i.e., the initial density $[\tilde{\rho}_0, H_1] = 0$, $\gamma = \exp(i\gamma_G) = 1$, then the GP is zero. The model discussed in Sec.II is the example.

In the above, the GP for degenerate ground state is discussed. As the parameters in the Hamiltonian changes, the ground state will have different distinct structures. We have obtained the relationship between GP and the initial ground states and the magnetic field, so that one can control the GP by modulating the magnetic field. The evolution period relies on the magnetic field.

From above, GP is dependent on the initial state and the Hamiltonian. For the model Eq.1, if one wish to detect the GP, one should try to improve the purity of the initial state and avoid the effect of decoherence. The external magnetic field can be helpful to control the geometric phase, which thus might aid in finding some applications in quantum computation.

The work was supported by NSFC No. 10674117 and 10405019.

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